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Solution of the Fokker–Planck Equation with Mixing of Angular Harmonics by Beam–Beam Charge Exchange

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A method for solving the linear Fokker-Planck equation with anisotropic beam–beam charge exchange loss is presented. The 2D equation is transformed to a system of coupled 1D equations which are solved iteratively as independent equations. Although isotropic approximations to the beam–beam losses lead to inaccurate fast ion distributions, typically only a few angular harmonics are needed to include accurately the effect of the beam–beam charge exchange loss on the usual integrals of the fast ion distribution. Consequently, the algorithm converges very rapidly and, in the absence of other strongly anisotropic processes, is much more efficient than a 2D finite difference method. A convenient recursion formula for the coupling coefficients is given and generalization of the method is discussed. © 1991 Academic Press, Inc.

1. INTRODUCTON

Intense neutral beam heating of low density plasmas [1–4] can produce a charge exchange loss rate between the injected fast neutrals and the trapped fast ions that approaches the thermalization rate. The anisotropic velocity distributions of the beam neutrals and fast ions together with the strong energy dependence of the charge exchange cross section produce a charge exchange loss rate with strong pitch angle dependence. When a Legendre expansion is used to represent the angular dependence of the distribution function [5, 6], the 1D equations for each Legendre harmonic are coupled by the anisotropic charge exchange loss. In the absence of any anisotropic processes (e.g., collisions with anisotropic particle distributions, acceleration by an electric field or magnetic compression, etc.) the 1D equations for each Legendre harmonic are independent and the system is very easily solved. In order to simplify the presentation only the coupling from beam–beam charge exchange is treated; presumably other processes could be handled in a fashion similar to that given below.

In spite of this complexity a solution based on Legendre harmonics is preferable to a 2D finite difference method because the fast ion particle density, current density, parallel and perpendicular pressures, beam–target, and beam–beam fusion reactivities are integral quantities which involve only low order harmonics; the first

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three or four usually suffice [6–8]. A finite difference algorithm with respectable accuracy, on the other hand, requires many angular grid points and correspondingly more calculation even if operator splitting is used to transform the 2D problem to a pair of 1D problems. In addition, the most efficient method of calculating the beam–beam fusion reactivity is based on Legendre harmonics [8]; a distribution based on a difference grid needs to be converted to a Legendre expansion to use it!

The general approach described in this paper can be more broadly applied [9]; any and all of the terms in the Fokker–Planck equation could have an angular dependence. A general nonlinear formulation [7] could be recast in this way. When the pitch angle domain is limited by loss cones, the basis functions are no longer Legendre polynomials and both the basis functions and the coupling coefficients must be determined numerically [7]. It is interesting to note that only a few harmonics are also needed in simulations of *simple* mirror machines [7], which are complementary to the example below in that they have near-perpendicular injection, loss cones, and anisotropic nonlinear Coulomb collision terms.

2. Coupled Equations for the Angular Harmonics

The usual approximate form (valid for $\exp(-(v/v_{\text{thi}})^2) \ll 1$ and $(v/v_{\text{the}})^2 \ll 1$) of the linear Fokker–Planck equation describing the fast ion velocity space distribution in a background plasma is

$$\frac{\partial f}{\partial t} = \frac{1}{\tau_{\text{se}} v^2} \frac{\partial}{\partial v} \left[(v^3 + v_{\text{c}}^3) f + \frac{1}{m_{\text{b}} v} (v^3 T_{\text{e}} + v_{\text{c}}^3 T_i) \frac{\partial f}{\partial v} \right] + \frac{v^3}{\tau_{\text{ps}} v_{\text{c}}^3} \frac{\partial}{\partial \zeta} (1 - \zeta^2) \frac{\partial f}{\partial \zeta} - \nu_{\text{cx}} f + S, \tag{1}$$

where $\zeta = v_{\|}/v$ and τ_{se}, v_{c}, and τ_{ps} are standard [10]. (Magnetic compression and electric field terms are neglected here.) The collision rates with Maxwellian background ions and electrons have been simplified by neglecting higher order corrections of order $\exp(-(v/v_{\text{thi}})^2)$ and $(v/v_{\text{the}})^2$, respectively.

The charge exchange loss rate can be divided into two parts

$$\nu_{\text{cx}} = n_{\text{o}}^{\text{th}} \langle \sigma v \rangle_{\text{cx}}^{\text{bt}} + n_{\text{o}}^{\text{b}} \langle \sigma v \rangle_{\text{cx}}^{\text{bb}}.$$

The "beam thermal" losses arise from reactions with the more or less isotropic thermal neutral gas arising from wall and volume sources; this term is not of specific interest here and is ignored below. The "beam–beam" loss rate is anisotropic because the velocity space distribution of neutrals in the injected beam is highly directional. In most circumstances there will be many beam–beam terms; one for each of the three energy components in each of the ion sources in a neutral beam system.

The beam–beam reactivity for a ring of fast ions with velocity space position of (v, θ) reacting with a delta-function distribution of fast neutrals at (v_o, θ_o) is

$$\langle \sigma v \rangle_{cx}^{bb} = \frac{1}{\pi} \int_0^{\pi} v_{rel} \sigma_{cx}(v_{rel})\, d\phi,$$

where $v_{rel}^2 = v^2 + v_o^2 - 2vv_o(\cos\theta \cos\theta_o + \sin\theta \sin\theta_o \cos\phi)$ and ϕ is the gyrophase angle relative to the component of the beam neutral velocity which is perpendicular to the magnetic field, $\cos\theta = \zeta$, and $\cos\theta_o = \zeta_o$.

For neutral and fast ion energies $\geqslant 30$ keV/amu $\langle \sigma v \rangle_{cx}^{bb}$ is a strong function of pitch angle (Fig. 1) because σ_{cx} is a strong function of v_{rel} [11]. The greatest anisotropy occurs for unidirectional injection with $|\zeta_o| \sim 1$ and $v \sim v_o$; $f(v, \zeta)$ is most anisotropic under the same conditions. For these reasons the test problem described below has parallel injection; the beam–beam fusion reactivity provides a sensitive scalar measure of changes in the distribution function.

By expanding the distribution function and ν_{cx} in Legendre polynomials

$$f(v, \zeta) = \sum_{m=0}^{M} f_m(v)\, P_m(\zeta),$$

$$\nu_{cx}(v, \zeta) = \sum_{n=0}^{N} \nu_n(v)\, P_n(\zeta),$$

multiplying Eq. (1) by $P_l(\zeta)$, and integrating over ζ, we obtain a coupled system of 1D equations for the angular harmonics of the distribution:

$$\frac{\partial f_l}{\partial t} = \frac{1}{\tau_{se} v^2} \frac{\partial}{\partial v} \left[(v^3 + v_c^3) f_l + \frac{1}{m_b v} (v^3 T_e + v_c^3 T_i) \frac{\partial f_l}{\partial v} \right]$$

$$- \frac{l(l+1)\, v^3}{\tau_{ps} v_c^3} f_l - \sum_{m=0}^{M} \sum_{n=0}^{N} A_{l,m,n} f_m \nu_n + S_l,$$

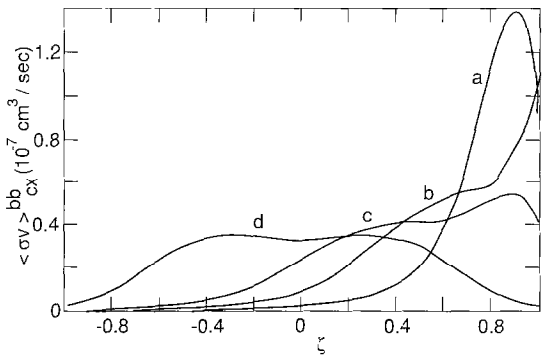

FIG. 1. The beam–beam charge exchange reactivity, $\langle \sigma v \rangle_{cx}^{bb}(v_o, \zeta)$, for 120 keV deuterium neutrals and fast ions where $\zeta_o =$ (a) 1.0; (b) 0.8; (c) 0.6; (d) 0.

for $l=0$ to M, where

$$S_l(v)=\frac{2l+1}{2}\int_{-1}^{1} S(v,\zeta)\,P_l(\zeta)\,d\zeta,$$

$$A_{l,m,n}=\frac{2l+1}{2}\int_{-1}^{1} P_l P_m P_n\,dx.$$

The coupling coefficients, $A_{l,m,n}$ are zero if $l+m+n$ is odd or if the sum of any pair of indices is less than the remaining index. J. C. Adams [12] derived a closed form expression for the coefficients

$$A_{l,m,n}=\frac{(2l+1)(m+n-l)!\,(l+n-m)!\,(l+m-n)!}{(2s+1)!\,(s-l)!^2\,(s-m)!^2\,(s-n)!^2},$$

where $2s=l+m+n$, but it may be more convenient to calculate them from the recursion relation

$$A_{l,m,n}=\begin{cases}\delta_{l,m}, & \text{for } n=0,\\ [(m+1)\,\delta_{l,m+1}+m\delta_{l,m-1}]/(2m+1), & \text{for } n=1,\\ \{(2n-1)[(m+1)\,A_{l,m+1,n-1}+mA_{l,m-1,n-1}]/(2m+1)\\ \quad -(n-1)\,A_{l,m,n-2}\}/n, & \text{for } n>1,\end{cases}$$

which follows from application of the identity

$$P_n=[(2n-1)\,xP_{n-1}-(n-1)\,P_{n-2}]/n$$

to the definition of $A_{l,m,n}$.

The diagonal elements, $A_{l,l,n}$, are easy to handle; only the elements with $l\neq m$ couple what would otherwise be independent equations for each f_l. The coupling is less burdensome than it appears because many elements of $A_{l,m,n}$ are zero and in practice N can be as small as 2 without serious loss of accuracy in the usual global properties of the fast ion distribution.

Ignoring the implicit dependence on f_l of the off-diagonal terms,

$$\sum_{m\neq l}^{M}\sum_{n=0}^{N} A_{l,m,n}f_m v_n,$$

we may consider them to be pseudo source terms. When solving the equation for each f_l, the off-diagonal terms are evaluated using the most recent versions of the f_m (which are initially zero). The iterative solution of this set of "independent" equations converges to the solution of the coupled set. Note that the coefficients of f_l in Eq. (2) are unchanged during the iterations; thus the coefficients in the tridiagonal difference equations are set up once and only the calculation of the "source" and the elimination and back-substitution are iterated. This is a very

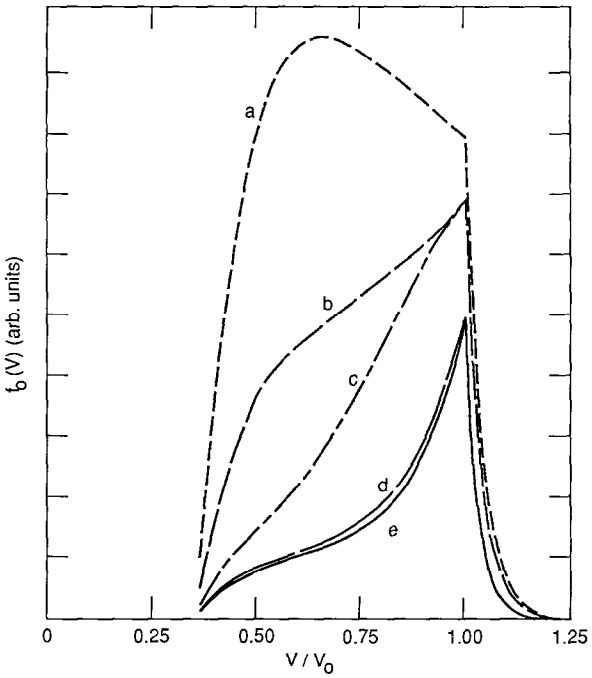

FIG. 2. The lowest harmonic of the fast ion distribution function, $f_0(v)$, for several treatments of the beam–beam charge exchange loss applied to the model problem described in the text: (a) no beam neutral density; (b) the (isotropic) beam–target charge exchange reactivity, $\langle \sigma v \rangle_{cx}^{bt}$, for cold neutral gas is used in place of $\langle \sigma v \rangle_{cx}^{bb}$. The first 11 harmonics of $f(v)$ together with 1, 3, and 11 harmonics of ν_{cx} are used for curves (c), (d), and (e), respectively.

convenient algorithm which usually converges very rapidly and involves only a modest change to existing codes.

The solution of an extreme situation is shown in Fig. 2. Even though the beam–beam charge exchange loss is very large, the iterative method converges. The boundary conditions are standard: $f(v_{thi}) = 0$ (thermalization sink) and zero particle flux at v_{max}. This example has 120 keV neutral deuterium injection with $\zeta_0 = 0.95$ in a deuterium background plasma at a density of $n_D = 5 \times 10^{19}\,m^{-3}$, with temperatures of 10 keV, and $Z_{eff} = 1.0$. A deuterium "beam" neutral density of $2 \times 10^{14}\,m^{-3}$ produces a deuterium source rate of $7 \times 10^{20}\,m^{-3}s^{-1}$. The source rate used in this test problem has been artificially reduced to insure that $n_{fast} \ll n_e$ so that the different methods of handling the beam–beam charge exchange loss do not affect n_e and, hence, the thermalization time scale. In this test problem $\tau_{se} = 1.5$ s, $E_c = 225$ keV, $\tau_{ps} = 1.1$ s, and $\nu_{0,1,2}^{-1}(v_0) = (0.22, 0.10, 0.12)$ s, respectively.

With such a large beam neutral density the beam–beam charge exchange loss has a dramatic effect on the steady-state fast ion distribution. The importance of the anisotropic part of the charge exchange loss is seen by comparing curve (c), which corresponds to using only ν_0, with curve (e), where the full angular dependence of

ν_{cx} is used. Both isotropic approximations—curves (b) and (c)—seriously underestimate the loss because the charge exchange reactivity has been averaged over pitch angle although most of the fast ions have a pitch angle near that of the neutrals.

The iterative procedure described above converged to a part in 10^6 in eight iterations. The fast ion particle density, pressure, and beam–beam fusion rate changed less than 5% when (M, N) were lowered from (11, 11) to (7, 3). When $|\zeta_0| \leqslant 0.8$, the accuracy is 5% or better with $(M, N) = (4, 3)$ or greater.

3. Discussion

The algorithm described in the previous section has worked very well in simulations of "supershots" in TFTR with convergence to the level of 10^{-6} occurring in three or four iteratons. The usual integrated properties of the distribution are typically changed by less than a percent when more than three harmonics are used to represent the charge exchange loss rate and the distribution. Beginning the iteration loop over the f_l with $l = M$ and working downward using the most recently calculated value of the f_m converges slightly faster than starting with $l = 0$ and working up in l. This occurs because the pitch angle scattering term in Eq. (2) grows as $l(l+1)$ and thus the high l harmonics are less sensitive to the beam–beam charge exchange and are closer to their final values on the first pass than the low l harmonics.

If many more harmonics are required, it is possible to use the techniques described by Karney [13] to vectorize more efficiently the elimination and back-substitution algorithm; the algorithm would also apparently be suitable for a parallel computer.

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